First-Order Methods for Large-Scale Market Equilibrium Computation

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Fisher Market: Equilibrium & Convex Optimization

A *Fisher market* consists of n buyers and m items. A market equilibrium (ME) is a set of allocations $x^* \in \mathbb{R}_+^{n\times m}$ and prices $p^* \in \mathbb{R}_+^m$ satisfying the follwing conditions [1, 2, 3]

- Each buyer i get the maximum utility out of budget B_i:

$$x_i^* \in \arg\max \left\{ u_i(x_i) : \langle p^*, x_i \rangle \le B_i,\ x_i \in \mathbb{R}_+^m \right\}.$$

- Each item j is sold, if price is nonzero:

$$\sum_{ij} x_{ij}^* \le 1 \text{ and } p_j^* > 0 \Rightarrow \sum_i x_{ij}^* = 1,\ \forall j.$$

Applications in ad auction, resource allocaiton and fair recommender systems require solving large-scale ME.

For many useful utility functions u_i, ME can be captured by convex programs [4, 5, 6].

- Linear: $u_i(x_i) = \langle v_i, x_i\rangle$ (perfectly complementary goods):

$$\text{(EG)} \quad \max \sum_i B_i \log u_i(x_i) \text{ s.t. } \sum_i x_{ij} \le 1,\ \forall j,\ x \ge 0.$$

- Quasi-linear (QL): $u_i(x_i) = \langle v_i - p, x_i\rangle$ (price deducted). We gave the following convex program, extending the Shmyrev's convex program for linear u_i [7, 8, 9], which is in buyers' *bids* b_{ij}:

$$\text{(S)} \quad \min_{b\ge 0} - \sum_{i,j:\, v_{ij}>0} (1+\log v_{ij}) b_{ij} + \sum_j p_j(b)\log p_j(b) \text{ s.t. } \sum_j b_{ij} \le B_i,\ \forall i, \text{ where } p_j(b) = \sum_i b_{ij}.$$

- Leontief: $u_i(x_i) = \min_{j\in J_i} \frac{x_{ij}}{a_{ij}}$ (perfectly substitute goods). EG also works for Leontief u_i.

These convex programs have objectives that are not strongly convex nor Lipschitz continuous. For the latter, we give new, tight bounds on equilibrium quantities utilizing properties of ME (assuming $\|B\|_1 = 1$, w.l.o.g.):

- Linear u_i: $B_i\|v_i\|_1 \le u_i^* \le \|v_i\|_1$, $\max_i \frac{v_{ij}B_i}{\|v_i\|} \le p_j^* \le 1$.
- QL u_i: $\max_i \frac{v_{ij}B_i}{\|v_i\|_1 + B_i} \le p_j^* \le \max_i v_{ij}$.
- Leontief u_i: $B_i\|a_i\|_\infty \le \langle a_i, p^*\rangle \le \|a_i\|_\infty$.

First-Order Methods

The following standard form captures many convex programs for ME:

$$\text{(P)} \quad \min_{x\in\mathcal{X}} f(x) = h(Ax) + \langle q, x\rangle$$

where h is μ-strongly convex with L-Lipschitz gradient, and $\mathcal{X}$ is a polyhedral set. First-order methods (FOM) can be used ot solve (P) [10]. Some of them are known to achieve linear convergence under various error-bound conditions, i.e., "weakened strong convexity" conditions (e.g., [11, 12]).

- Projected Gradient (PG): $x^{t+1} = \Pi_{\mathcal{X}}(x^t - \gamma_t \nabla f(x))$.
- Mirror Descent (MD): $x^{t+1} = \arg\min_{x\in\mathcal{X}} \langle \nabla f(x^t), x - x^t\rangle + \gamma_t D(x\|x^t)$.
- Frank-Wolfe (FW): $w^t \in \arg\min_{w:\,\text{a vertex of } \mathcal{X}} \langle \nabla f(x^t), w\rangle,\ x^{t+1} = x^t + \gamma_t(w^t - x^t)$.

FOM for ME

We establish *linear convergence* of PG with a practical linesearch for general convex optimization problems $f(x) + g(x)$ satisfying the *Proximal-PŁ* condition, a receintly proposed error-bound condition [11].

With this and our new bounds on equilibrium quantities (see upepr left box), we establish *linear convergence* of PG (with a static stepsize or linesearch) for solving (EG) and (S), via first reformulating them into (P).

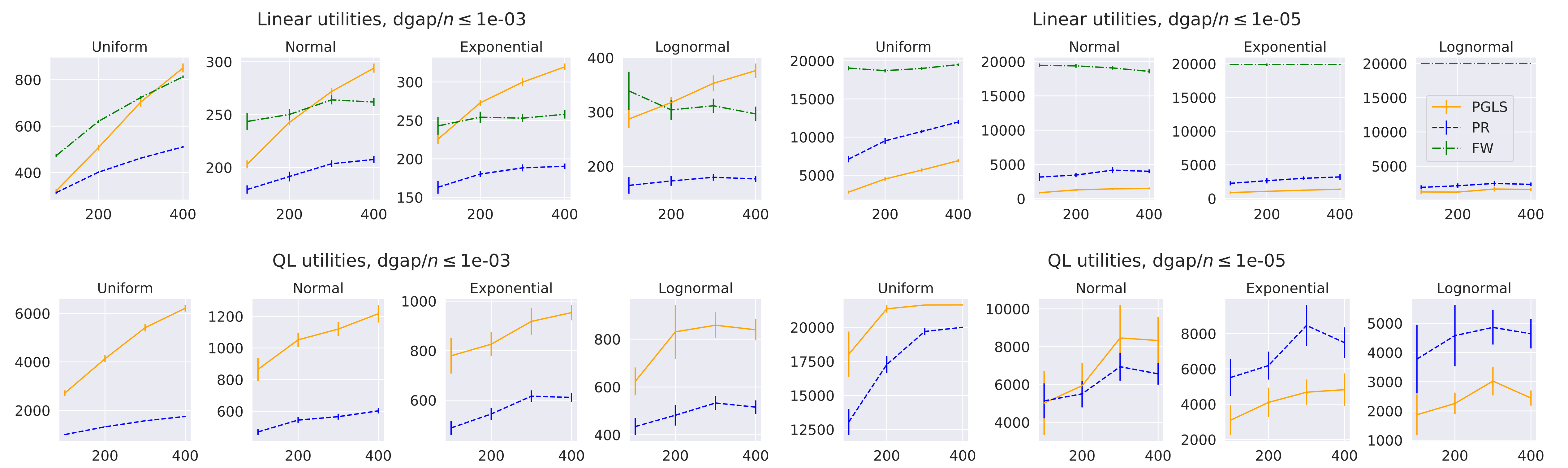
For QL u_i, adopting the analysis in [8], we show that mirror descent applied to (S) achieves a nonstandard last-iterate $1/T$ convergence, where φ is the objective:

$$D(p(b^t)\|p^*) \le \varphi(b^t) - \varphi^* \le \frac{\log(m+1)}{t}.$$

It also yields explicit updates similar to the *Proportional Response Dynamics* (PR) algorithm for finding ME under linear u_i [13, 8]. These updates are interpretable and highly scalable. At time t,

- Buyers submit their bids $b_{ij}^t \to$ item prices given by $p_j^t = \sum_i b_{ij}^t \to$ each buyer is allocated $x_{ij}^t = b_{ij}^t / p_j^t$.
- Bids and leftover budgets are updated via $b_{ij}^{t+1} = B_i \cdot \frac{v_{ij}x_{ij}^t}{\sum_\ell v_{i\ell}x_{i\ell}^t + \delta_\ell^t}$ and $\delta_i^{t+1} = B_i \cdot \frac{\delta_i^t}{\sum_\ell v_{i\ell}x_{i\ell}^t + \delta_\ell^t}$.

Experiments



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