Temperature Control for Langevin Diffusions

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Langevin Diffusion for Global Non-convex Optimization

- Global non-convex optimization: $\min_{x\in\mathbb{R}^d} f(x)$.
- Gradient descent may converge to a suboptimal local minimum.
- Continuous Langevin algorithm or overdamped Langevin diffusion:

$$dX(t) = \underbrace{-f_x(X(t))dt}_{\text{Gradient descent}} + \underbrace{\sqrt{2\beta(t)}dW(t)}_{\text{Brownian noise}}, \quad t \geq 0 .$$

- Problem: design *endegeneous state-dependent* temperature $\beta(t)$.
- The classical optimal control of such a problem is of the bang-bang type, which is overly sensitive to any errors.

Exploratory Control Formulation

- We study the entropy-regularized stochastic relaxed control problem:

$$V(x) = \inf_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} f(X^\pi(t))dt - \lambda \int_0^\infty e^{-\rho t} \int_U -\pi(t,u)\ln\pi(t,u)dudt\right],$$

  - $\int_U -\pi(t,u)\ln\pi(t,u)du$: the differential entropy of the probability density function $\pi(t,\cdot)$ of the randomized temperature,
  - U: the range of the temperature,
  - $\lambda > 0$: a weight parameter measures the strength of regularization,
  - X^π is governed by

$$dX^\pi(t) = -f_x(X^\pi(t))dt + \sigma(\pi(t))dW(t), \quad X^\pi(0) = x,$$

where $\sigma(\pi) := \sqrt{\int_U 2u\pi(u)du}$.

Results

- Optimal feedback law is truncated exponential distribution:

$$\bar{\pi}(u;x) = \frac{1}{Z(\lambda, v_{xx}(x))}\exp\left(-\frac{1}{\lambda}[\mathrm{tr}(v_{xx}(x))u]\right), \quad u \in U,$$

where v satisfies the equation

$$-\rho v(x) - f_x(x)\cdot v_x(x) + f(x) - \lambda\ln(Z(\lambda, v_{xx}(x))) = 0, \quad x \in \mathbb{R}^d.$$

- Optimal state process

$$dX^*(t) = -f_x(X^*(t))dt + h(X^*(t))dW(t),$$

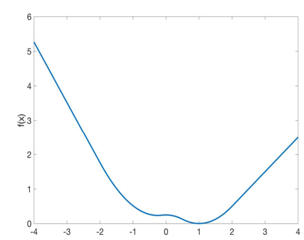
where

$$h(x) := \sqrt{\frac{2}{Z(\lambda, v_{xx}(x))}\int_U u\exp\left(-\frac{1}{\lambda}[\mathrm{tr}(v_{xx}(x))u]\right)du}.$$

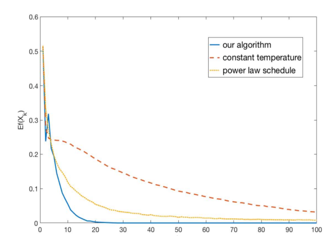
Algorithm and Numerical Example

Comparison of three algorithms on a baseline non-convex function

- Langevin algorithm with constant temperature
- Langevin algorithm with power-law temperature schedule
- Our algorithm with state-dependent temperature



(a) A double-well function



(b) Comparison

Research Support

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